Statistics of Maximal Independent Sets in Grid-like Graphs

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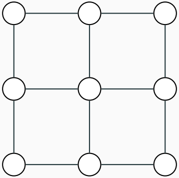
Preliminaries

Definition (Independent Sets)

A set $I \subseteq V(G)$ is called an **independent set** if no two vertices in I are adjacent.



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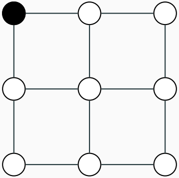

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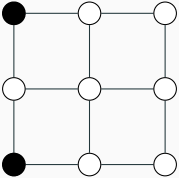

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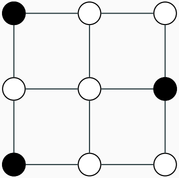

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- Enumeration of $\mathrm{MIS}(G)$ is known to be $\#P$-complete
- Miller and Muller (1960):

$$\mathrm{MIS}(G) \leq \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \pmod 3 \\ 4.3^{n/3-1} & \text{if } n \equiv 1 \pmod 3 \\ 2.3^{n/3} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

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- Moon and Moser (1965), Erdős (1966): Bounding $g(n) :=$ the maximum number of different sizes of MIS's

$$n - \log n - H(n) - O(1) \leq g(n) \leq n - \log n$$

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Let $V_i := \{(i,j) : 1 \leq j \leq m\}$. A graph G is **grid-like** provided that

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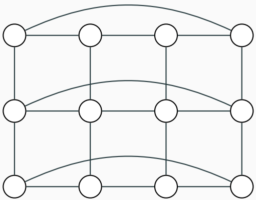
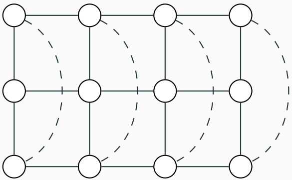
In this paper, $H \in \{P_m, C_m\}$.

Preliminaries

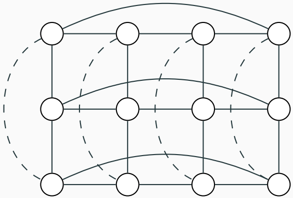
There are four particular grid-like graphs that we study. They are pictured below for $m = 3$ and $n = 4$:



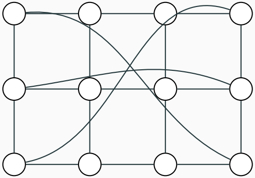
Fat Cylinder: $FC_{m \times n}$



Thin Cylinder: $TC_{m \times n}$



Torus: $T_{m \times n}$



Möbius Strip: $M_{m \times n}$

Contributions

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- Enumerating non-isomorphic MIS's for small m.
- Finding the average size of MIS's for small m.

Framework

Transfer Matrices

Golin et al. (2005) surveyed a set of enumeration problems on grid graphs, grid-cylinders, and grid-tori of fixed height, which can be modeled by the *transfer matrix approach*, including

- Hamiltonian Cycles
- Perfect Matchings
- Spanning Trees
- Cycle Covers

On such a grid-like graph to count $S(m, n)$ objects, the method finds vectors a, b and a square matrix A such that

$$|S(m, n)| = a^\top A^n b$$

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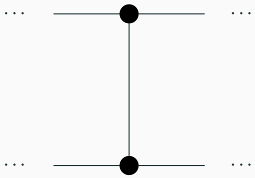

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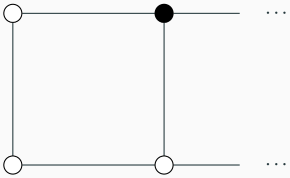
2. The first and last columns of M must include 1 vertex.



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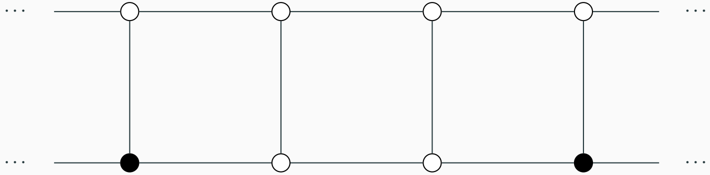

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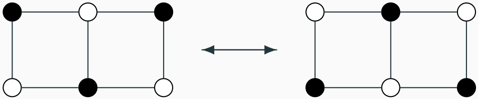
3. For every two adjacent columns, there is at least one vertex in M.
4. M has a unique dual, formed by reflecting its choice of vertices over the horizontal axis between the two rows.



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By (1) and (2), an MIS of $\mathrm{MIS}(G_{2\times n})$ contains exactly one of the vertices in the last column. Consider the sets which contain $(n,2)$.

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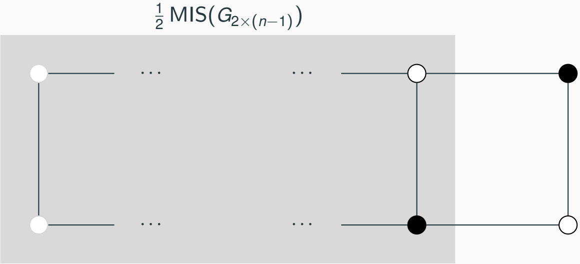

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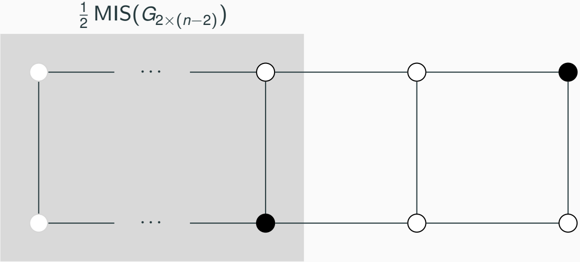

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Thus, by (4),

$$|\mathsf{MIS}(G_{2\times n})| = 2\left(\frac{1}{2}|\mathsf{MIS}(G_{2\times(n-1)})| + \frac{1}{2}|\mathsf{MIS}(G_{2\times(n-2)})|\right)$$

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With the initial conditions

$$|\mathsf{MIS}(G_{2\times 1})| = 2 \quad , \quad |\mathsf{MIS}(G_{2\times 2})| = 2$$

$$|\mathsf{MIS}(G_{2\times n})| = 2F_n$$

States

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Definition (State of local structure)

Let H be the local structure of G. A **state** of H is an ordered pair (I, D) in which

1. I is an independent set of H such that $H[V(H) \setminus N[I]]$ is 2-colorable;
2. D, the **deficit**, is a color class of a 2-coloring of $H[V(H) \setminus N[I]]$

We define $U(I) := V(H) \setminus N[I]$ to be the *uncovered set* of a state (I, D).

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Definition (State orderings)

State (I', D') **follows** state (I, D) or provided that

1. $I \cap I' = \emptyset$
2. $D \subseteq I'$
3. $D' = U(I') \setminus I$.

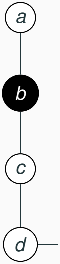
State Definition Example

In this state,

$H = P_4$

$I = \{b\}$

$D = \{d\}$



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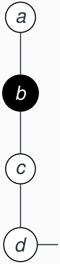

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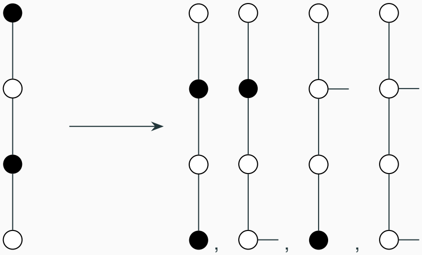

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Let $S(H)$ be the set of states of H. Let $M(H)$ be the **map digraph** of G with

$$V(M(H)) := S(H) \qquad , \qquad E(M(H)) := \{\overrightarrow{s_1 s_2} : (s_1, s_2) \in S(H)^2, s_1 \vdash s_2\}$$

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Definition (Ticket Digraph)

Let T be the **ticket digraph** of G with

$$V(T) := S(H) \qquad , \qquad E(T) := \{\overrightarrow{s_1 s_2} : (s_1, s_2) \in S(H)^2, \text{ an } \mathit{MIS} \text{ can start in state } s_1 \text{ and end in state } s_2\}$$

Transfer Matrix Application

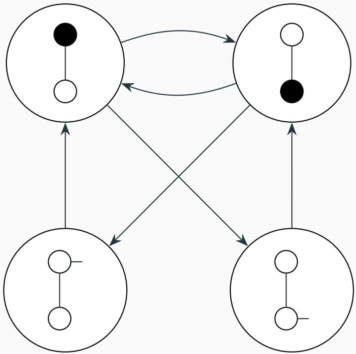
Theorem (Transfer Matrix Application)

Let $A_{M(H)}$ be the adjacency matrix of $M(H)$ and A_T be the adjacency matrix of T. Then,

$$\tau(n) = |\,\mathrm{MIS}(G)| = A_T \cdot A_{M(H)}^{n-1}$$

Map and Ticket Digraph Example

Let G have local structure P_2. The map digraph of G is



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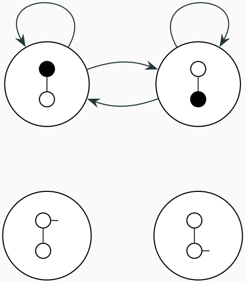

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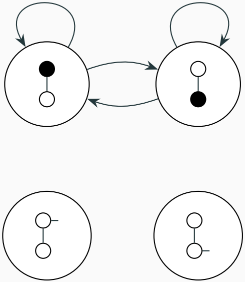

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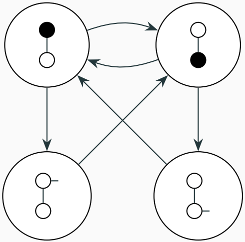

Global Cyclic Structure

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- For each $v \in M$, $d^+(v) = 2^k$ for some $k \in \mathbb{N}_0$.
- $\Delta^+(M(P_m)) = 2^{\lceil \frac{m}{2} \rceil}$, $\Delta^+(M(C_m)) = 2^{\lfloor \frac{m}{2} \rfloor}$

Enumeration

$\tau(n)$ is a linear recurrence

Theorem ($\tau(n)$ is a linear recurrence)

Let (M, T) be the auxiliary digraphs of G on k. Let A_M and A_T be the adjacency matrices of M and T respectively. Let $f(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0$ be the characteristic polynomial of A_M. Then, τ satisfies the recurrence

$$\tau(n) = -a_{k-1}\tau(n-1) - \ldots - a_1\tau(n-k+1) - a_0\tau(n-k)$$

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Proof

- By the Cayley-Hamilton Theorem, A_M satisfies the linear recurrence given by f.

$$A_M^n = -a_{k-1}A_M^{n-1} - \ldots - a_1 A_M^{n-k+1} - a_0 A_M^{n-k}$$

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Theorem ($\tau(n)$ is a linear recurrence)

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$$\tau(n) = -a_{k-1}\tau(n-1) - \ldots - a_1\tau(n-k+1) - a_0\tau(n-k)$$

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Recall that $|\mathrm{MIS}(G_{2\times n})| = 2F_n$. Because $\tau(n) = \tau(n-1) + \tau(n-2)$, τ grows exponentially with rate $\varphi = \frac{1+\sqrt{5}}{2}$. More specifically,

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Theorem (Existence of c, r)

Let $m \in \mathbb{N}$. The sequences $|\mathrm{MIS}(G_{m\times n})|, |\mathrm{MIS}(FC_{m\times n})|, |\mathrm{MIS}(TC_{m\times n})|, |\mathrm{MIS}(T_{m\times n})|, |\mathrm{MIS}(M_{m\times n})|$ as functions of n all obey linear recurrences. Moreover, for each sequence $\tau(n)$, there exists real numbers $c > 0$ and $r > 1$ such that

$$\lim_{n\to\infty} \frac{\tau(n)}{c \cdot r^n} = 1$$

Proof:

Theorem (Perron (1907), Frobenius (1912))

Let A be a primitive square matrix. Then, A has a Perron-Frobenius eigenvalue r, i.e. an eigenvalue equal to its spectral radius, such that the left and right eigenspaces of r are generated by single strictly positive vectors $\vec{w}^\top$ and $\vec{v}$ respectively. Moreover

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Theorem (Main Result)

Let $H = P_m$ or C_m and let φ be a graph automorphism of H. Let r be the principle eigenvalue of the map digraph M of H. Then,

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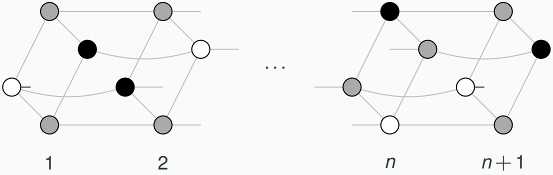

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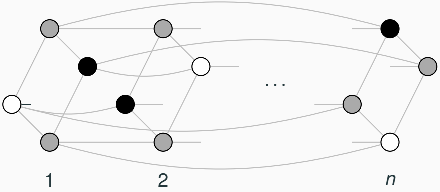

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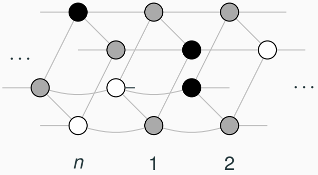

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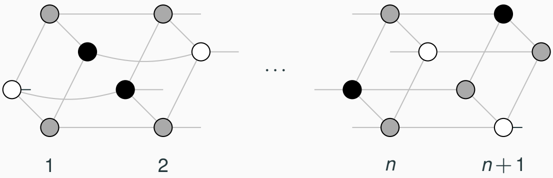

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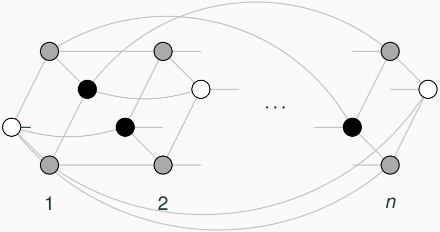

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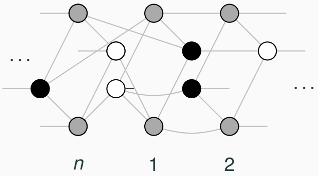

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- Thus, only the ticket digraph of G is affected.

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Proof:

- Let A_{T_φ} be the induced ticket digraph. It is a permutation matrix with 1s at $(i, \phi(i))$. Sufficient to show that $c = 1$.

$$c = A_{T_\varphi} \bullet \frac{\vec{v}\vec{w}^\top}{\vec{w}^\top \vec{v}} = \sum_{1\le i\le m} \frac{\vec{v}_i \vec{w}_{\varphi(i)}}{\vec{w}^\top \vec{v}}$$

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- What does $\vec{w}_{\varphi(i)}^{\top}$ look like?
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  - We can show that $w^{\top} P$ is a left eigenvector of A corresponding to the principle eigenvalue.

$$(w^{\top} P)A = w^{\top}(PA) = (w^{\top} A)P = rw^{\top} P = r \cdot w^{\top} P$$

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Two elements I, I' are *isomorphic* if there exists a graph automorphism $\varphi : G \to G$ with $\varphi(I) = I'$ and *non-isomorphic* if no such φ exists. Denote the set of non-isomorphic MISs on G by $\mathsf{NIMIS}(G)$.

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- In general, we can use the group of symmetries of our graph to act on the set of MISs. The number of distinct orbits of $\mathrm{MIS}(G)$ counts $|\mathrm{NIMIS}(G)|$.

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